Lecture 2A: Graph Theory I

UC Berkeley EECS 70
Summer 2022
Tarang Srivastava

Announcements!

430
Soda Hall

- Read the Weekly Post
- Tarang's OH 4-6p in Woz Lounge (Zoom also–same link as lecture)
    - First 30 minutes for conceptual question
    - Last 90 minutes for reading Note 5 together and question about the note
    - Will not prioritize HW questions. Use regular OH for that.
- **HW 2** and **Vitamin 2** have been released, due **Thu** (grace period Fri)
- We are adding a bit more OH support, but also work on the HW early

  Use Piazza
- Throughout this lecture **definitions** will be underlined

Undirected Simple Graph Definitions

An undirected simple **graph** $G = (V, E)$ is defined by

1. A set V of **vertices**. Sometimes we may call it a **node**.
2. A set E of **edges** $\{u, u\}$

Where edges in E are of the form $\{u, v\}$ for u, v *in* V and $u \neq v$.

A graph being **simple** here means no parallel edges

A graph being **undirected** means there's no direction to the edges

Examples:

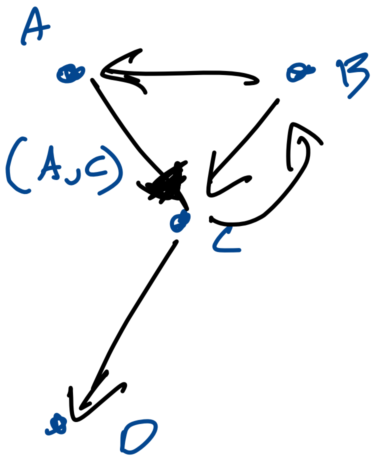
Directed Graph Definitions

tuple

Edges in a <u>**directed graph**</u> are defined as (u, v). That is, the order of the vertices matters. Therefore, $(u, v) \neq (v, u)$.

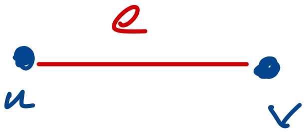
(u, v)

Examples:



Edge and Degree Definitions

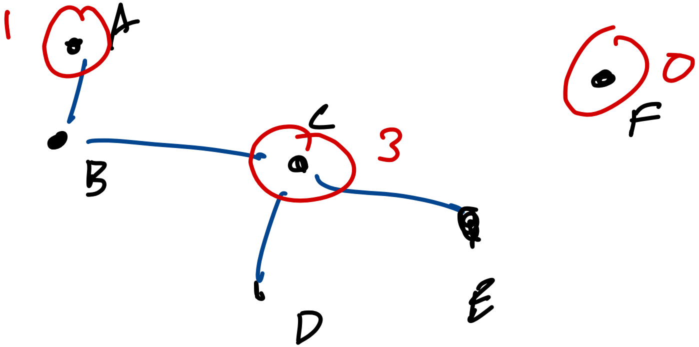
Given an edge $e = \{u, v\}$ we say



on

- e is **incident** to u and v
- u and v *are* **neighbors**
- u and v *are* **adjacent**
- The **degree** of a vertex v is the number of incident edges
    - $deg(v) = |\{v \text{ in } V \mid \{u, v\} \text{ in } E\}|$

Examples:



Summary Questions I

How many nodes in this graph? 11

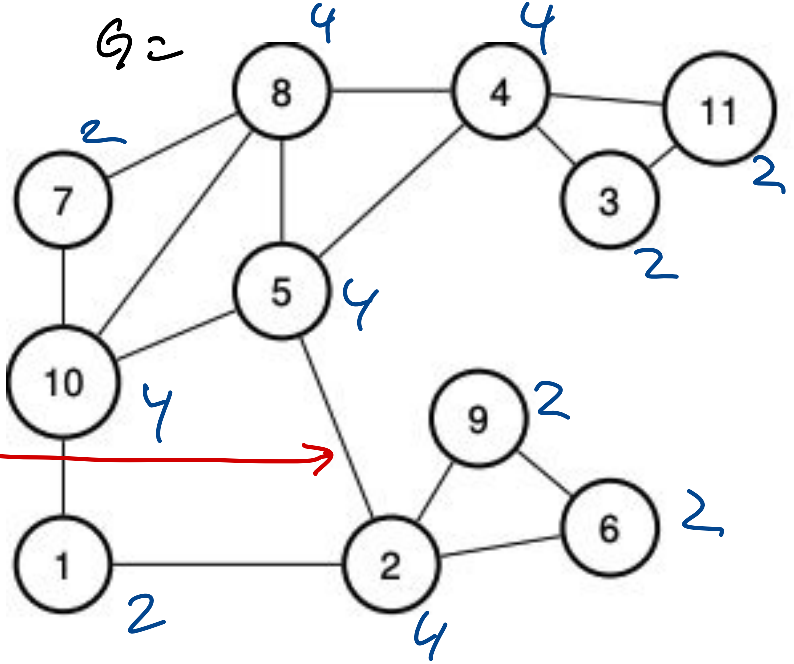
How many edges? 16

Which vertex has the max degree? 5, 8, 2, 10, 4

Which vertex has the min degree? 6, 11, 3, 9, 7, 1

Which vertices is this edge incident on? 2 and 5

What is the sum of the degrees? 32



Handshake Lemma

$A = \{1, 2, 3\}$

$|A| = 3$

$|\cdot|$ ← size

"cardinality"

Lemma: The sum of the degree of all the vertices is equal to $2|E|$

Proof: Proceed by induction on $|E| = m$

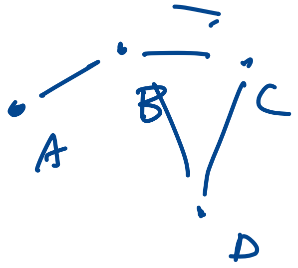
Base Case: $m = 0$. A graph has no edges if all the vertices are isolated (i.e. no neighbors) thus each vertex is degree 0

$0 + \ldots + 0 = 2(0)$ ✓

Ind Hyp: Assume claim holds for $m = k$ edges, ... sum of degrees is $2k$

Ind Step: Consider an arbitrary graph G with $k+1$ edges. Remove any edge from G. The new graph has k edges, and by the inductive hypothesis sum of degrees is $2k$. Then adding back the edge we add 1 degree to each incident vertex. Thus sum of degrees is now $2k + 2 = \underbrace{2(k+1)}_{\#\text{ edges}}$ as desired □

Path, Cycles, Walks and Tours



Deals with Vertices (though may imply things about edges):

A, B, C   simple

<u>Path:</u> A sequence of vertices in G, generally with no repeated vertices.

<u>Cycle:</u> A path in G where the only repeated vertex is the first one and last one.

Deals with Edges (though may imply things about vertices):

{A, B}, {B, C}   A → B → C

<u>Walk</u>: Is a sequence of edges with possible repeated vertex or edges.

<u>Tour</u>: A walk that starts and ends at the same vertex.

<u>Eulerian walk</u>: A walk where each edge is visited exactly once.

<u>Eulerian tour</u>: An Eulerian walk that starts and ends at the same vertex

Summary Questions II

Give an example of length 3 cycle? 4, 3, 11, 4

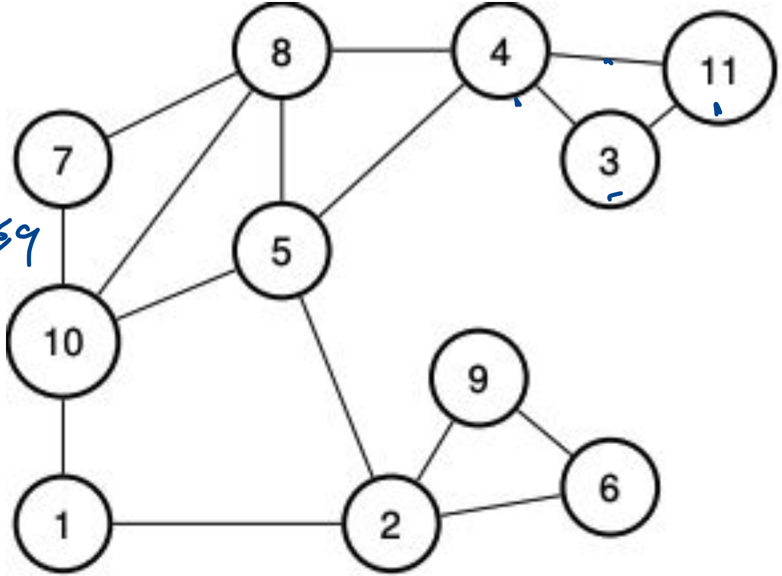
Give an example of a path from 2 to 8? 2, 5, 8

4, 3, 11, 5

Give the longest simple path? , 8, 7, 10, 1, 2, 6, 9

~~How many connected components are there?~~

Give an example of length 4 tour? 7 → 10 → 5 → 8

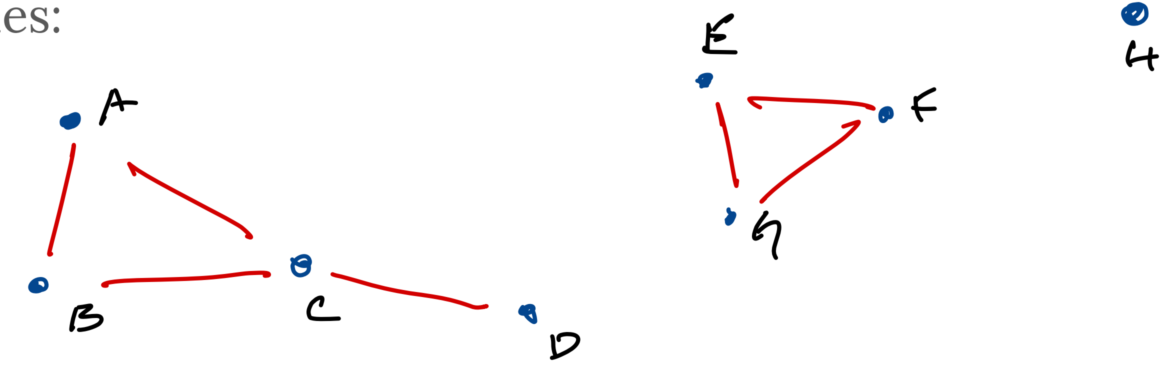


Connectivity

A graph G is said to be **connected** if there exists a path between any two vertices.

$V_1 = \{A, B, C, D\}$
$V_2 = \{E, F, G\}$
$V_3 = \{H\}$

Examples:



Any graph always consists of a collections of **connected components.** A connected component is a set of vertices in the graph that are connected.

see CS170 SCC

Eulerian Tours

<u>Eulerian walk</u>: A walk where each edge is visited exactly once.

<u>Eulerian tour</u>: An Eulerian walk that starts and ends at the same vertex

Theorem: A undirected graph G has an Eulerian ~~tour~~ walks iff G is even degree, and connected.

⟺

every vertex has even degree

except two vertices that are odd

Proof: *in the notes*

⟹

~~⟸~~

Summary Questions III

Is there an Eulerian Tour and if so provide a tour?

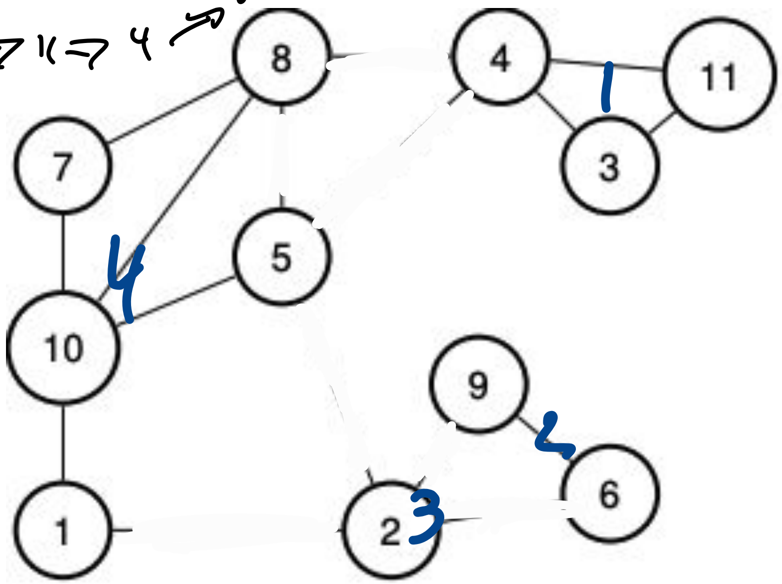
6 → 2 → 1 → 10 → 7 → 8 → 10 → 5 → 4 → 3 → 11 → 4 → 8 → 5 → 2 → 9 → 6

Why? every vertex has even degree & connected

How many connected components now? 1

Connected components now? 1

What about now? 4



Graph Proof

98

Counter example

~~Fa~~lse Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$

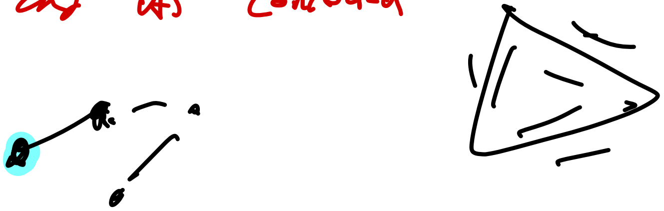
Base Case: There is only one graph with a single vertex and it has degree 0. Thus, vacuously true.

Inductive Hypothesis: Assume the claim is true for ~~some~~ $n = k$ $n \geq 1$

Inductive Step: We prove the claim is also true for $n + 1$. Consider an undirected graph with n vertices and each has degree greater than 1. ~~By the inductive hypothesis, this graph is connected.~~ Now add one more vertex x to obtain a graph with $(n + 1)$ vertices.

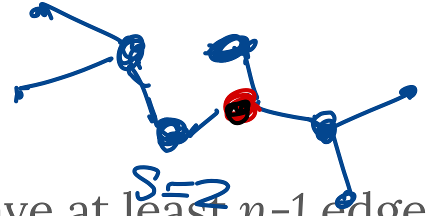
Since, the previous graph was connected, and x is connected to some node y then there's a path between x and any other vertex through y, since by definition there's a path from y to any other vertex. Thus, the graph is connected.

n+1 vertices and its connected



Minimum Edges for Connectivity

$P(n) \Rightarrow P(n+1)$



Theorem: Any connected graph with n vertices must have at least n-1 edges

Induction on vertices $n = |V|$

Base Case: $n=1$ 0 edges $0 \le (1)-1$

Ind Hyp: Assume claim holds for $1 \le n \le k$ $\quad S = \deg(v)$

Ind Step: Consider a connected graph G with $n = k+1$ vertices. Remove an arbitrary vertex v. Removing v suppose creates S connected components, where S at most $\deg(v)$. By strong induction each connected component has $k_1 - 1$ edges, $k_2 - 1$ edges ... $k_S - 1$ edges. Adding back v we get to add back S edges

$$\underbrace{k_1 + k_2 + \dots + k_S}_{} \underbrace{-1-1-1 \dots -1}_{S}$$

$|E| \ge k - S$

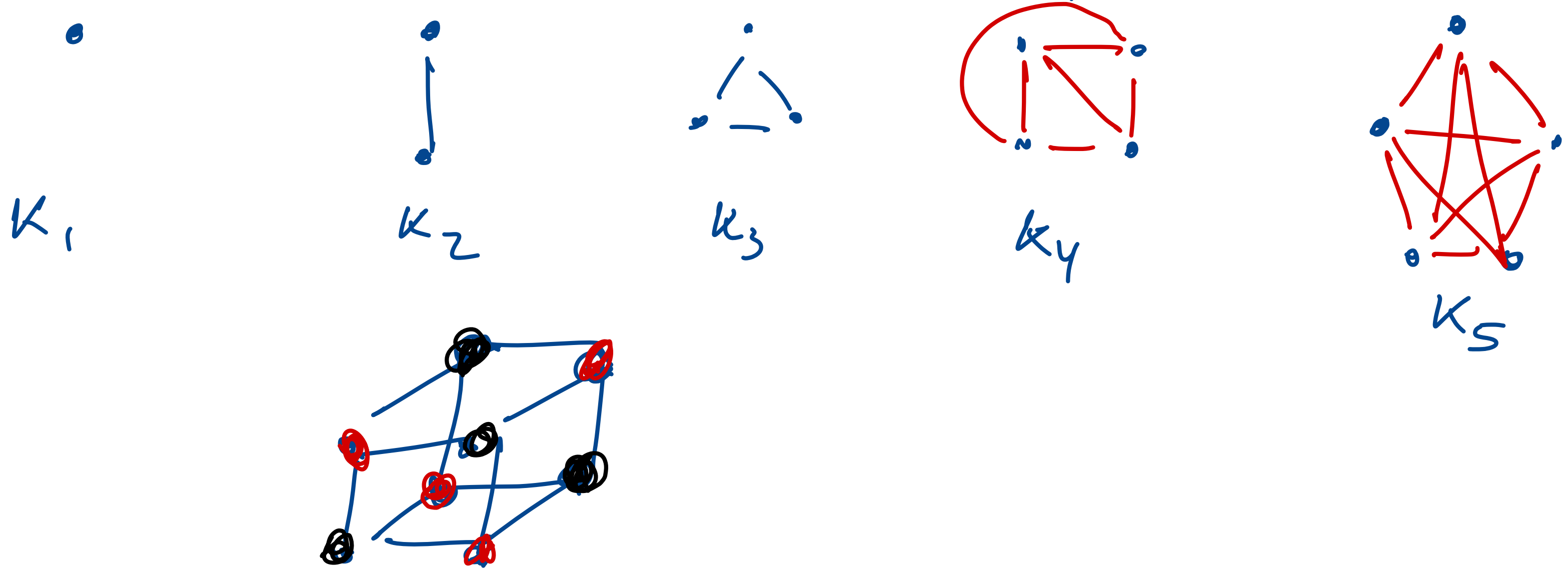
$(k+1)-1$

$|E| \ge k$ as done

Complete Graphs

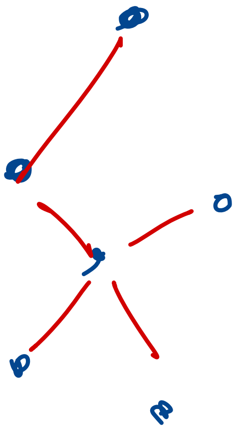
A graph G is **complete** if it contains the maximum number of edges possible.

Examples:



Trees

G



The following definitions are all equivalent to show that a graph G is a **tree**.

1. G is connected and contains no cycles
2. G is connected and has n-1 edges (where $n = |V|$)
3. G is connected, and the remove of any single edge disconnects G
4. G has no cycles, and the addition of any single edge creates a cycle

Tree Definitions are Equivalent

Theorem: For a connected graph G it contains no cycles iff it has n-1 edges.
Proof:

Tree Definitions are Equivalent (cont.)

Theorem: For a connected graph G it contains no cycles iff it has n-1 edges.

Bipartite Graphs

A graph G is **bipartite** if the vertices can be split in two groups (L or R) and edges only go between groups.

Examples: